Existence and optimality necessary conditions for general stochastic mean-field control problems

Khaled Bahlali†     Meriem Mezerdi ‡  Brahim Mezerdi§

November 18, 2014

Abstract

This paper is concerned with optimal control problems for systems governed by mean-field stochastic differential equation, in which the control enters both the drift and the diffusion coefficient. In these problems the coefficients depend not only on the state but also on on the distribution of the state process. These equations are obtained as the limits of some interacting particle systems and play an important role in game theory. We study the relaxed model which is a natural extension of the initial problem, in the sense that the two problems have the same value function, in which the admissible controls are measure-valued processes. We prove that the natural relaxed state process is a stochastic differential equation driven by an orthogonal martingale measure, whose quadratic variation is exactly the relaxed control. We prove the existence of an optimal relaxed control, by using tightness properties and Skorokhod theorem. Moreover we give necessary conditions for optimality in terms of two adjoint processes in the form of Peng’s maximum principle. It should be noted that there is no way to obtain this maximum principle with just the first order adjoint process.

Key words: Mean-field stochastic differential equation, relaxed control, martingale measure, approximation, tightness, weak convergence, adjoint process, maximum principle.

MSC 2010 subject classifications, 93E20, 60H30.

1 Introduction

In this paper we bring the necessary corrections to a paper published in Automatica.....by Adel Chala.

This paper is concerned by mean field-control problems, where the state process is governed by a stochastic differential equation of mean-field type (MFSDE in short) given by:

\[
\begin{align*}
    dX_t &= b(t, X_t, E(\psi(X_t)), u_t)dt + \sigma(t, X_t, E(\phi(X_t)), u_t)dW_t \\
    X(0) &= X_0.
\end{align*}
\]

The objective of the controller is to minimize over a set of admissible controls, the following mean-field cost functional

\[
J(u) = E\left( \int_0^T h(t, X_t, E\varphi(X_t), u_t)dt + g(X_T, E\chi(X_T)) \right).
\]

Note that in this kind of control problems, the coefficients depend not only on the state process but also on its distribution. MFSDEs are obtained as mean square limits of interacting particle systems of the form:

\[
dX_t^{i,n} = b(t, X_t^{i,n}, 1/n \sum_{j=1}^n \psi(X_t^{i,n}), u_t)dt + \sigma(t, X_t^{i,n}, 1/n \sum_{j=1}^n \Phi(X_t^{i,n}), u_t)dW_t
\]
When \( n \) goes to infinity, it is proved in [26], in the linear case, that \( X^{i,n}_t \) converges to \( X^i_t \), where all the processes \( X^i_t (i = 1, \ldots) \), are independent copies of the same process, called the non linear process or the McKean-Vlasov process, which is the unique solution of the MFSDE (1.1). We refer to [19], to the general case of a non linear dependence of the coefficients upon the process and its distribution and the driving process is a general Lévy process.

This kind of approximation result is called "propagation of chaos", which roughly speaking, says that when the number of players tends to infinity, then the equations defining the evolution of the players could be replaced by a single equation, called the McKean-Vlasov equation. This mean-field equation, represents in some sense the average behavior of the infinite number of players.

Since the publication of the pioneering papers Lasry and Lions [21] and Huang, Caines and Malhamé [17], mean-field control theory has raised a lot of interest, motivated by applications to game theory and mathematical finance. One can refer in particular to, Lasry–Lions [25–27], Gueant et al. [17], Huang et al. [21,22], Buckdahn et al. [13], Andersson–Djehiche [1], Cardaliaguet [14], Carmona–Delarue [15], Bensoussan et al. [10], and the references therein. See also the recent book by Bensoussan et al.

A typical example of mean field control problems is the so-called continuous-time Markowitz's mean-variance portfolio selection problem, see [2, 12, 28, 31]. The main drawback, when dealing with mean field problems, is that the Bellmann principle does not hold and as a consequence, there is no HJB equation for the value function. For this kind of problems, the stochastic maximum principle, provides a powerful tool to solve them, see [2, 6, 11, 22, 23, 27, 28]. The SMP gives necessary optimality conditions in terms of the maximization of some Hamiltonian and an adjoint process which is the solution of a backward SDE of mean field type, see [7, 9].

In this paper, we are interested by the existence of an optimal control. It is well known that in simple deterministic and stochastic examples that an optimal strict control may fail to exist, in the absence of suitable convexity conditions. The idea is then to introduce the class of relaxed controls, known also in the deterministic literature as Young measures, sliding or chattering controls, see [1]. By enlarging the space of admissible controls so as it contains measures, we do not change the value function, that is the infimum of the cost functional among strict controls is equal to the infimum among relaxed controls. The set of relaxed controls, when equipped with stable convergence, is a compact separable metrizable space. In fact, the fundamental idea behind the introduction of relaxed controls is the completion of the space of ordinary controls. In this completion the ordinary controls will be identified as Dirac measures. Existence results for classical stochastic control problems can be found in [14] [13, 15] [16, 24, 5]. See also [3, 4, 8] for systems driven by backward and forward-backward SDEs.

We prove the existence of an optimal relaxed control, for control problems driven by non linear MFSDEs. The proof is based on tightness properties of the underlying processes and Skorokhod selection theorem. We prove that the relaxed state process is no longer driven by a Brownian motion, but by a continuous orthogonal martingale measure. By doing this, in fact we correct a paper by Chala.

Moreover, due to the compactness of the action space, we show that the relaxed control could be chosen among the so-called sliding controls, which are convex combinations of Dirac measures. As a consequence and under some Filippov convexity condition, the relaxed control is shown to be strict.

The second main result is a stochastic maximum principle for the relaxed control problem. These are a set of necessary conditions for a relaxed control to be optimal, obtained via the first and second order adjoint processes. The proof is based on the chattering lemma, some stability properties of the state and adjoint processes with respect to the control variable. We then apply Ekeland's variational principle to derive necessary conditions for a sequence of nearly optimal strict controls. The result is then obtained by a passage to the limit. This result may be seen as an extension of Peng's maximum principle to mean field controls and also extends the result of Bu Dj Li to relaxed controls and the result of BMM to the case where the diffusion coefficient depends explicitly on the control variable.

Note that Chala has considered the same mean field control problem, but in the relaxed state process he replaces simply the drift and diffusion coefficients by their relaxed counterparts. As it will be shown in a simple counter example, this is not the right relaxed model. The reason is that the stochastic integral does not behave as a Lebesgue integral. In fact one has to relax the quadratic variation of the stochastic integral part of the equation, rather than the stochastic integral itself. Roughly speaking the idea is to relax the generator of the process in place of the equation itself.
2 Existence of optimal relaxed controls

2.1 Controlled mean field stochastic differential equations

Let $(W_t)$ is a $d$-dimensional Brownian motion, defined on a probability space $(\Omega, \mathcal{F}, P)$, endowed with a filtration $(\mathcal{F}_t)$, satisfying the usual conditions. Let $A$ be some compact metric space called the control set.

We study the existence of optimal controls for systems driven non linear mean field SDEs of the form

\[
\begin{align*}
\{ & dX_t = b(t, X_t, E(\Psi(X_t)), u_t)dt + \sigma(t, X_t, E(\Phi(X_t), u_t)dW_t \\
& X_0 = x
\end{align*}
\]  

(2.1)

and the cost functional over the time interval $[0, T]$ is given by

\[
J(U) = E \left( \int_0^T h(t, X_t, E(\varphi(X_t), u_t) \right) dt + g(X_T, E(\lambda(X_T))
\]  

(2.2)

where $b, \sigma, l, h$ and $\psi$ are given functions. The control variable $u_t$, is a measurable, $\mathcal{F}_t-$ adapted process with values in the action space $A$.

The following assumptions will be in force throughout this paper.

$$(H_1)$$ Assume that

\[
\begin{align*}
b : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times A & \longrightarrow \mathbb{R}^d \\
\sigma : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times A & \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d \\
\Psi : \mathbb{R}^d & \longrightarrow \mathbb{R}^d \\
\Phi : \mathbb{R}^d & \longrightarrow \mathbb{R}^d 
\end{align*}
\]  

(2.3)

are bounded continuous functions and there exists $K > 0$ such that for any pairs $(x_1, y_1)$ and $(x_2, y_2)$ in $\mathbb{R}^d \times \mathbb{R}^d$:

\[
\begin{align*}
|b(t, x_1, y_1, u) - b(t, x_2, y_2, u)| & \leq K (|x_1 - x_2| + |y_1 - y_2|) \\
|\sigma(t, x_1, y_1, u) - \sigma(t, x_2, y_2, u)| & \leq K (|x_1 - x_2| + |y_1 - y_2|) \\
|\Psi(x_1) - \Psi(x_2)| & \leq K (|x_1 - x_2|) \\
|\Phi(x_1) - \Phi(x_2)| & \leq K (|x_1 - x_2|)
\end{align*}
\]  

(2.4)

$$(H_2)$$ Assume that

\[
\begin{align*}
h : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times A & \longrightarrow \mathbb{R} \\
g : \mathbb{R}^d \times \mathbb{R}^d & \longrightarrow \mathbb{R} \\
\varphi : \mathbb{R}^d & \longrightarrow \mathbb{R} \\
\lambda : \mathbb{R}^d & \longrightarrow \mathbb{R}^d 
\end{align*}
\]  

(2.5)

are bounded continuous functions and $h$ is $K$-Lipschitz continuous in the variables $(x, y)$, that is there exists $K > 0$ such that for any pairs $(x_1, y_1)$ and $(x_2, y_2)$ in $\mathbb{R}^d \times \mathbb{R}^d$:

\[
|h(t, x_1, y_1, u) - h(t, x_2, y_2, u)| \leq K (|x_1 - x_2| + |y_1 - y_2|)
\]  

(2.6)

Proposition 2.1. Under assumption $(H_1)$ the MFSDE (2.1) has a unique strong solution. Moreover for each $p > 0$ we have $E(|X_t|^p) < +\infty$.

Proof.

Let us define $\overline{b}(t, x, \mu, a)$ on $[0, T] \times \mathbb{R}^d \times M_1(\mathbb{R}^d) \times \mathbb{R}^k$ and $\overline{\sigma}(t, x, \mu, a)$ on $[0, T] \times \mathbb{R}^d \times M_1(\mathbb{R}^d)$ by

\[
\begin{align*}
\overline{b}(t, x, \mu, a) &= b(t, x, \int \Psi(x) d\mu(x), a) \\
\overline{\sigma}(t, x, \mu, a) &= \sigma(t, x, \int \Phi(x) d\mu(x), a)
\end{align*}
\]  

(2.7)
where \( M_1(\mathbb{R}^d) \) denotes the space of probability measures in \( \mathbb{R}^d \).

According to Proposition 1.2 in [19] it is sufficient to check that \( \overline{b} \) and \( \overline{\sigma} \) are Lipschitz in \((x, \mu)\). Indeed since the coefficients \( b \) and \( \sigma \) are Lipschitz continuous in \( x \), then \( \overline{b} \) and \( \overline{\sigma} \) are also Lipschitz in \( x \). Moreover one can verify easily that \( \overline{b} \) and \( \overline{\sigma} \) are also Lipschitz continuous in \( \mu \), with respect to the Wasserstein metric

\[
d(\mu, \nu) = \inf \left\{ \left( E^Q |X - Y|^2 \right)^{1/2} ; Q \in M_1(\mathbb{R}^d \times \mathbb{R}^d), \text{ with marginals } \mu, \nu \right\}
\]

\[
= \sup \left\{ \int h d(\mu - \nu) ; |h(x) - h(y)| \leq |x - y| \right\},
\]

(2.7)

where \( M_1(\mathbb{R}^d \times \mathbb{R}^d) \) is the space of probability measures on \( \mathbb{R}^d \times \mathbb{R}^d \). Note that the second equality is given by the Kantorovich-Rubinstein theorem [20]. Since the mappings \( b \) and \( \Psi \) in the the MFSDE are Lipschitz continuous in \( x \) we have

\[
\left| b(\ldots, \int \Psi(x) d\mu(x), \ldots) - b(\ldots, \int \Psi(x) d\nu(x), \ldots) \right|
\leq K \left| \int \Psi(x) d(\mu(x) - \nu(x)) \right|
\leq K^* d(\mu, \nu)
\]

(2.8)

Similar arguments can be used for \( \sigma \).

Using similar techniques as in Proposition 1.2 in [19], it holds that for each \( p > 0 \), \( E(|X_t|^p) < +\infty \).

2.2 The relaxed control problem

If we do not assume convexity conditions, an optimal control may fail to exist in the set \( \mathcal{U}_{ad} \) of strict controls even in deterministic control problems. To be convinced let us consider the following example.

The problem is to minimize \( J(u) = \int_0^1 (X^u(t))^2 dt \) over the set \( \mathcal{U}_{ad} \) of open loop controls, that is, measurable functions \( u : [0, 1] \rightarrow \{-1, 1\} \), where \( X^u(t) \) denotes the solution of \( dX^u(t) = u(t) dt, X(0) = 0 \).

We have \( \inf_{u \in \mathcal{U}_{ad}} J(u) = 0 \).

Consider the following sequence of controls:

\[
u_k(t) = (-1)^k \text{ if } \frac{k}{n} \leq t \leq \frac{k+1}{n}, 0 \leq k \leq n - 1.
\]

Then clearly \(|X^{\nu_k}(t)| \leq 1/n \) and \(|J(u_k)| \leq 1/n^2 \) which implies that \( \inf_{u \in \mathcal{U}_{ad}} J(u) = 0 \). There is however no control \( \tilde{u} \) such that \( J(\tilde{u}) = 0 \). If this would have been the case, then for every \( t \), \( X^{\tilde{u}}(t) = 0 \), which implies that \( u_k = 0 \), which is impossible.

The problem is that the sequence \((u_n)\) has no limit in the space of strict controls. This limit, if it exists, will be the natural candidate for optimality. If we identify \( u_n(t) \) with the Dirac measure \( dt \delta_{u_n(t)}(du) \), then \( (dt \delta_{u_n(t)}(du))_n \) converges weakly to \((1/2)dt (\delta_{-1} + \delta_1)(du)\). This suggests that the set of strict controls is too narrow and should be embedded into a wider class with a reacher topological structure for which the control problem becomes solvable. The idea of relaxed control is to replace the \( A \)-valued process \((u_t)\) with a \( M_1(A) \)-valued process \((\mu_t)\), where \( M_1(A) \) is the space of probability measures equipped with the topology of weak convergence.

For the relaxed model to be a true extension of the original control problem, must satisfy the following two conditions:

i) The value functions of the original and the relaxed control problems must be equal.

ii) The relaxed control must have an optimal solution.
2.2.1 The canonical space of the set of relaxed controls

Let $\mathcal{M}_1(A)$ be the space of probability measures on the control set $A$. Let $\mathcal{V}$ be the space of measurable transformations $\mu : [0, T] \rightarrow \mathcal{M}_1(A)$, then $\mu$ can be identified as a nonnegative measure on the product $[0, T] \times A$, by putting for $C \in \mathcal{B}([0, T])$ and $D \in \mathcal{B}(A)$

$$\mathcal{P}(C \times D) = \int_C \mu_t(da)dt$$

$\mathcal{P}$ can be extended uniquely to an element of $\mathcal{M}_+([0, T] \times A)$ the space of Radon measures on $[0, T] \times A$, equipped with the topology of stable convergence. This topology is the weakest topology such that the mapping

$$\mathcal{P} \rightarrow \int_0^T \int_A \phi(t, a).\mathcal{P}(dt, da)$$

is continuous for all bounded measurable functions $\phi$ which are continuous in $a$.

Equipped with this topology, $\mathcal{M}_+([0, T] \times A)$ is a compact separable metrizable space. Therefore $\mathcal{V}$ as a closed subspace of $\mathcal{M}_+([0, T] \times A)$ is also compact (see El Karoui, Hauss-Lepeltier, [?]) for more details.

Notice that $\mathcal{V}$ can be identified as the space of positive Radon measures on $[0, T] \times A$, whose projections on $[0, T]$ coincide with Lebesgue measure.

Let us define the Borel $\sigma-$field $\mathcal{V}$ as the smallest $\sigma-$field such that the mappings

$$\int_0^T \int_A \phi(t, a).\mu_t(da)dt$$

are measurable, where $\phi$ is a bounded measurable function which is continuous in $a$.

Let us also introduce the filtration $(\mathcal{V}_t)$ on $\mathcal{V}$, where $\mathcal{V}_t$ is generated by $\{1_{[0,t]}\mu, \mu \in \mathcal{V}\}$.

Definition 2.2. A relaxed control on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ is a random variable $\mu$ with values in $\mathcal{V}$ such that $\mu(\omega, t, da)$ is progressivly measurable with respect to $(\mathcal{F}_t)$ and such that for each $t$, $1_{[0,t]}\mu$ is $\mathcal{F}_t-$measurable.

2.2.2 The relaxed state process

A natural question arises is the following: what is the natural SDE associated with a relaxed control. Note that in the deterministic case or in the stochastic case where only the drift is controlled, one has just to replace in equation [ ] the drift

$$b(t, X_t, E(\Psi(X_t)), u_t) \quad \text{by} \quad \int_A b(t, X_t, E(\Psi(X_t)), a)\mu_t(da).$$

See Fl, Mez-Bah, etc...

Now we are in a situation where both the drift and diffusion coefficient are controlled. Let us try a direct relaxation of the original equation as in the deterministic case. That is replace

$$b(t, X_t, E(\Psi(X_t)), u_t) \quad \text{by} \quad \int_A b(t, X_t, E(\Psi(X_t)), a)\mu_t(da)$$

and $\sigma(t, X_t, E(\Psi(X_t)), u_t) \quad \text{by} \quad \int_A \sigma(t, X_t, E(\Psi(X_t)), a)\mu_t(da)$

The "relaxed" control problem will be governed by the MFSDE

$$\begin{cases}
\frac{dX_t}{dt} = \int_X b(t, X_t, \Psi(X_t), a)\mu_t(da)dt + \int_A \sigma(t, X_t, \Psi(X_t), a)\mu_t(da)dW_t \\
X_0 = x
\end{cases}$$

This model has been considered in Chala for mean field controls problems and by Bahlali, and NU Ahmed for classical control problems.

As it will be shown in the sequel, this model does not fullfill the requirements of a true relaxed model. The reason is that the value function of the original control problem is not equal to the value function of the relaxed problem. Let us consider the following example:

Consider a control problem governed by the following SDE without mean field terms:
\[
\begin{aligned}
&dX_t = u_t dW_t \\
&X_0 = x
\end{aligned}
\]

where the control \( u \in \mathcal{U}_{ad} \) the set of measurable functions \( u : [0,1] \to [-1,1] \)

The relaxed model will be governed by the equation

\[
\begin{aligned}
&dX_t = \int_A a\mu_t(da) dW_t \\
&X_0 = x
\end{aligned}
\]

Consider once more the sequence

\[
u_n(t) = (-1)^k \text{ if } \frac{k}{n} \leq t \leq \frac{(k+1)}{n}, \quad 0 \leq k \leq n - 1.
\]

It is clear that \( X^n_t = \int_0^t u_n(s) dW_s \) is a continuous martingale with quadratic variation

\[
\langle X^n, X^n \rangle_t = \int_0^t u^2_n(s) ds = t.
\]

We know that the sequence of corresponding relaxed controls \( dt.\delta_{\nu_n(t)}(da) \) corresponding to the sequence \( (u_n) \) converges weakly to \( \mu^* = (1/2)dt(\delta_{-1} + \delta_1)(da) \). Let \( X^* \) be the corresponding relaxed state process corresponding to the limit \( \mu^* \), then

\[
X^*(t) = \int_0^t \int_A (1/2)(\delta_{-1} + \delta_1)(da) dW_t = 0.
\]

It is obvious that the sequence of state processes \( (X^n_t) \) does not converge in any way to \( X^*_t \).

This implies in particular that the state process is not continuous in the control variable and as a byproduct, the value functions of the strict and "relaxed" control problems are not equal. Moreover even if the set of relaxed controls is compact, there is no mean to prove existence of an optimal control for this model.

In fact the model considered in Chala, Bahlali and NU Ahmed is wrong. In using it these authors have shown in particular that they can obtain a stochastic maximum principle, when \( \sigma \) is controlled by using only the first order adjoint process. As it was proved in the example above, this is completely false.

**What is the right relaxed state process?**

The reason why the proposed model is wrong, is that the stochastic integral part does not behave as a Lebesgue integral. In fact, one should relax the quadratic variation of the martingale part, which is a Lebesgue integral.

In the relaxed model, the quadratic variation process \( \int_0^t \sigma^2(t, X_t, E(\Phi(X_t), u_t)) dt \) corresponding to a strict control \( u_t \) can now be relaxed and replaced by \( \int_0^t \int_A \sigma^2(t, X_t, E(\Phi(X_t), a)) \mu_t(da) dt \), which is more natural than relaxing the stochastic integral. This in fact is equivalent to the relaxation of the infinitesimal generator.

Note that Equation (MFSDE) has a weak solution is equivalent to the martingale problem

\[
f(X_t) - f(X_0) - \int_0^t Lf(s, X_s, u_s) ds \text{ is a } \mathcal{F}_t \text{-martingale},
\]

for each \( f \in C^2_b \), for each \( t > 0 \), where \( L \) is the infinitesimal generator associated with Eq.(MFSDE),

\[
Lf(t, x, u) = \frac{1}{2} \sum_{i,j} \left( a_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} (t, x, u) + \sum_i \left( b_i \frac{\partial f}{\partial x_i} \right) (t, x, u),
\]

\[
b = b(t, x, (\Phi, \nu), a) \text{ and } a_{i,j} = \sigma^2(t, x, (\Phi, \nu), a).
\]

In fact the right relaxation is to relax the generator \( L \). Then the relaxed martingale problem becomes:
\[ f(X_t) - f(X_0) - \int_0^t Lf(s, X_s, a) \mu_a(da)ds \text{ is a } P\text{-martingale} \]

for each \( f \in C_b^2 \), for each \( t > 0 \).

The natural question now is what is the stochastic differential equation corresponding to the relaxed martingale problem? The answer is given by the following theorem.

**Theorem 2.3.** 1) Let \( P \) be the solution of the Martingale Problem (3). Then \( P \) is the law of a d-dimensional adapted and continuous process \( X \) defined on an extension of the space \((\Omega, F, F_t)\) and solution of the following SDE starting at \( x \):

\[
dX_t = \int \mathbb{A} b(t, X_t, E(\Psi(X_t))) a) \mu_a(da)dt + \int \mathbb{A} \sigma(t, X_t, E(\Phi(X_t)), a) M(da, dt),
\]

where, \( M = (M^k)_{k=1}^d \) is a family of d-strongly orthogonal continuous martingale measures, each having intensity \( \mu_a(da)dt \).

2) If the coefficients \( b \) and \( \sigma \) are Lipschitz in \( x, y \), uniformly in \( t \) and \( a \), the SDE (2.6) has a unique pathwise solution.

**Proof.**

1) See El Karoui Méléard Theorem ..or Watkins Theorem 3.1 for the situation where there is no mean field terms. In fact adding mean field terms does not add additional difficulties.

2) The coefficients being Lipschitz continuous, following the same steps as in [19], it is not difficult to prove that Equation 2.9 has a unique solution such that for each \( p > 0 \) we have \( E(|X_T|^p) < +\infty \). The fact that the SDE is driven by a martingale measure does not add particular difficulties.

\[
J(\mu) = E \left( \int_0^T \int A h(t, X_t, E(\varphi(X_t)), a) \mu_t(da) dt + g(X_T, E(\lambda(X_T))) \right).
\]

The coefficients being Lipschitz continuous, following the same steps as in [19], it is not difficult to prove that Equation 2.9 has a unique solution such that for each \( p > 0 \) we have \( E(|X_T|^p) < +\infty \). The fact that the SDE is driven by a martingale measure does not add particular difficulties.

**2.2.3 Approximation of the relaxed model**

We show in this section that the strict and the relaxed control problems have the same value function. This is based on the chattering lemma and the stability of the state process with respect to the control variable.
Lemma 2.5. (Chattering lemma)

i) Let \((\mu_t)\) be a relaxed control. Then there exists a sequence of adapted processes \((u^n(t))\) with values in \(A\), such that the sequence of random measures \((\delta_{u^n}(da))\) converges in \(\mathcal{V}\) to \(\mu_t(da)\), \(P - a.s.\)

ii) For any \(g\) continuous in \([0,T] \times \mathcal{M}_1(A)\) such that \(g(t,.)\) is linear, we have \(P - a.s\)

\[
\lim_{n \to +\infty} \int_0^t g(s, \delta_{u^n})ds = \int_0^t g(s, \mu_s)ds \text{ uniformly in } t \in [0,T].
\] (2.11)


Proposition 2.6. If we denote \(M^n(t,B) = \int_0^t f_B \delta_{u^n}(da) dW_s\), then for every bounded predictable process \(\varphi : \Omega \times [0,T] \times A \to \mathbb{R}\), such that \(\varphi(\omega, t, .)\) is continuous, we have

\[
E\left[ \left( \int_0^t \varphi(t, a) M^n(da) - \int_0^t \varphi(t, a) M^n(da) \right)^2 \right] \to 0 \text{ as } n \to +\infty.
\]

Proof. See Méléard [1], page 18, for the proof.

Proposition 2.7. 1) Let \(X^u, X^n\) be the solutions of state equation (2.9) corresponding to \(\mu\) and \(u^n\), where \(\mu\) and \(u^n\) are defined as in the chattering lemma. Then

\[
\lim_{n \to \infty} E\left[ \sup_{0 \leq t \leq T} |X^u_t - X^n_t|^2 \right] = 0.
\] (2.12)

2) Let \(J(u^n)\) and \(J(\mu)\) the expected costs corresponding respectively to \(u^n\) and \(\mu\). Then there exists a subsequence \((u^{n_k})\) of \((u^n)\) such that \(J(u^{n_k})\) converges to \(J(\mu)\).

Proof. 1) Let \(\mu\) a relaxed control and \((dt \delta_{u^n}(da))\) the sequence of atomic measures associated to the sequence of strict controls \((u^n)\), as in the last Lemma. Let \(X^u, X^n\) the corresponding state processes. Then

Let us write the strict state process associated with \(u^n\) in a relaxed form. Let \(dt \delta_{u^n}(da)\) be the relaxed control corresponding to \(u^n\) and \(M^n(t,B) = \int_0^t \int_B \delta_{u^n}(da) dW_s\), then \(X^n\) satisfy the following MSDE

\[
\begin{align*}
\begin{cases}
\d X_t^n = \int_A b(t, X^n_t, \Psi(X^n_t)), a) \delta_{u^n_t}(da) dt + \int_A \sigma(t, X_t, \Psi(X^n_t)), a) M^n(da)
\end{cases}
\end{align*}
\]

Where we have

\[
|X^n_t - X^u_t| \leq \left| \int_0^t \int_A a(s, X^n_s, \Psi(X^n_s), u) \mu_s(u) du ds - \int_0^t \int_A a(s, X^n_s, \Psi(X^n_s), u) \delta_{u^n}(da) ds \right|
\]

\[
+ \left| \int_0^t \int_A b(s, X^n_s, \Psi(X^n_s), u) \mu_s(u) du ds - \int_0^t \int_A b(s, X^n_s, \Psi(X^n_s), u) \delta_{u^n}(da) ds \right|
\]

\[
+ \left| \int_0^t \int_A a(s, X^n_s, \Psi(X^n_s), u) \delta_{u^n}(da) ds - \int_0^t \int_A a(s, X^n_s, \Psi(X^n_s), u) \delta_{u^n}(da) ds \right|
\]

\[
+ \left| \sup_{s \leq t} |v, X^n_s, \Psi(X^n_s)| M^n(da) - \int_0^t \int_A a(s, X^n_s, \Psi(X^n_s), u) \delta_{u^n_t}(da) ds \right|
\]

\[
+ \left| \sup_{s \leq t} |v, X^n_s, \Psi(X^n_s)| M^n(da) - \int_0^t \int_A a(s, X^n_s, \Psi(X^n_s), u) \delta_{u^n_t}(da) ds \right|
\]

Then by using Burkholder-Davis-Gundy inequality for the martingale part and the fact that all the functions in equation (2.9) are Lipschitz continuous, it holds that

\[
E\left( \sup_{0 \leq t \leq T} |X^n_t - X^u_t|^2 \right) \leq K \left[ \int_0^T E\left( \sup_{0 \leq s \leq T} |X^n_s - X^u_s|^2 \right) dt + \varepsilon_n \right]
\] (2.13)
where $K$ is a nonnegative constant and

\[
\varepsilon_n = E \left( \sup_{0 \leq t \leq T} \left| \int_0^t \int_A b(s, X_s, E(\Psi(X_t), u)) \mu_s(da) ds - \int_0^t \int_A b(s, X_s, E(\Psi(X_s), a)) \delta_{u}^n(da) ds \right| \right)
\]

(2.14)

\[
+ E \left( \sup_{0 \leq t \leq T} \left| \int_0^t \int_A \sigma(s, X_s, E(\Psi(X_t), a)) \mu_s(da) ds - \int_0^t \int_A \sigma(s, X_s, E(\Psi(X_s), a)) \delta_{u}^n(da) ds \right| \right)
\]

(2.15)

By using the chattering lemma and the Lebesgue dominated convergence theorem, it holds that $\lim_{n \to +\infty} \varepsilon_n = 0$. We conclude by using Gronwall’s Lemma.

2) Property 1) implies that the sequence $(X^n_t)$ converges to $X_t$ in probability uniformly in $t$, then there exists a subsequence $(X^{n_k}_t)$ which converges to $X_t$, $P$-a.s uniformly in $t$. We have

\[
|J(u^n) - J(\mu)| \leq E \left[ \int_0^T \left| h(t, X^n_t, E(\varphi(X^n_t), a)) - h(t, X_t, E(\varphi(X_t), a)) \right| \delta_{u^n}(da) dt \right]
\]

\[
+ E \left[ \int_0^T \left| \int_A h(t, X_t, E(\varphi(X_t), a) \delta_{u^n}(da) dt - \int_0^T \int_A h(t, X_t, E(\varphi(X_t), a) \mu(t, da) dt \right| \right]
\]

\[
+ E \left[ |g(X^n_T, E(\lambda(X^n_T))) - g(X_T, E(\lambda(X_T)))| \right]
\]

It follows from the continuity and boundedness of the functions $h, g, \varphi$ and $\lambda$ with respect to $x$ and $y$, that the first and third terms in the right hand side converge to 0. The second term in the right hand side tends to 0 by the weak convergence of the sequence $\mu^n$ to $\mu$, the continuity and the boundedness of $h$ in the variable $a$. We use the dominated convergence theorem to conclude.

**Remark 2.8.** As a consequence of Proposition 2.6, it holds that the infimum among relaxed controls is equal to the infimum among strict controls. This is equivalent to say that the value functions for the strict and relaxed control problems are the same.

### 2.3 Existence of optimal relaxed controls

The following theorem which is the main result of this section, extends [7, 14, 13, 15] to systems driven by mean field SDEs with controlled diffusion coefficient.

**Theorem 2.9.** Under assumptions (H1), (H2), there exist an optimal relaxed control.

The proof is based on some auxiliary Lemmas and will be given later.

Let $(\mu^n)_{n \geq 0}$ be a minimizing sequence, that is $\lim_{n \to \infty} J(\mu^n) = \inf_{\mu \in K} J(\mu)$ and let $X^n$ be the unique solution of our MFSDE associated with $\mu^n$:

\[
\begin{aligned}
X^n_0 &= x + \int_0^T \int_A b(s, X^n_s, E(\Psi(X^n_t), u)) \mu_s^n(du) ds + \int_A \sigma(t, X_t, E(\Psi(X_t)), a) M^n(dt, da), \\
X^n_0 &= x.
\end{aligned}
\]

(2.16)

The proof of the main result consists in proving that the sequence $(\mu^n, M^n, X^n)$ is tight and then show that we can extract a subsequence which converges in law to a process $(\tilde{\mu}, \tilde{W}, \tilde{X})$, which satisfies the same MFSDE. To finish the proof we prove that the sequence of cost functionals $(J(\mu^n))_n$ converges to $J(\tilde{\mu})$ which is equal to $\inf_{\mu \in K} J(\mu)$ and then $(\tilde{\mu}, \tilde{W}, \tilde{X})$ is optimal.

**Lemma 2.10.** The sequence of distributions of the relaxed controls $(\mu^n)_n$ is relatively compact in $\mathbb{V}$.

**Proof.** The relaxed controls $\mu^n$ are random variables on the space $\mathbb{V}$ which is compact. Then by applying Prohorov’s theorem yields that the family of distributions associated to $(\mu^n)_{n \geq 0}$ is tight then it is relatively compact. ■
Lemma 3.4 The family of martingale measures \( (M^n_n)_{n \geq 0} \) is tight in the space \( C_{S'} = C([0,1], S') \) of continuous functions from \([0,1]\) into \( S'\), the topological dual of the Schwartz space \( S \) of rapidly decreasing functions.

**Proof.** The martingale measures \( M^n_n, n \geq 0 \), can be considered as random variables with values in \( C_{S'} = C([0,1], S') \) (see Mitoma [20]). By [20] Lemma 6.3, it is sufficient to show that for every \( \varphi \) in \( S \) the family \( (M^n_n(\varphi), n \geq 0) \) is tight in \( C([0,T], \mathbb{R}^d) \) where \( M^n_n(\omega,t,\varphi) = \int_A \varphi(a) M^n_n(\omega, t, da) \). Let \( p > 1 \) and \( s < t \). By the Burkholder-Davis-Gundy inequality we have

\[
E \left( |M^n_n(\varphi) - M^n_n(\varphi)|^{2p} \right) \leq C_p E \left[ \left( \int_0^t |\varphi(a)|^2 \mu^n_n(da) dt \right)^p \right] 
= C_p E \left[ \left( \int_0^t |\varphi(u^n_n)|^2 dt \right)^p \right] 
\leq C_p \sup_{a \in A} |\varphi(a)|^{2p} |t-s|^p 
\leq K_p |t-s|^p,
\]

where \( K_p \) is a constant depending only on \( p \). That is the Kolmogorov tightness criteria in \( C([0,T], \mathbb{R}^d) \) is fulfilled. Hence the sequence \( (M^n_n(\varphi)) \) is tight.

Lemma 3.5 The family of processes \( (X^n_n)_{n \geq 0} \) is tight in the space \( C([0,T], \mathbb{R}^d) \)

**Proof.** Let \( p > 1 \) and \( s < t \). Using usual arguments from stochastic calculus and the boundness of the coefficients \( b \) and \( \sigma \), it is easy to show that

\[
E \left( |X^n_n - X^n_n|^p \right) \leq C_p |t-s|^p
\]

which yields the tightness of \( (X^n_n, n \geq 0) \) in \( C([0,T], \mathbb{R}^d) \).

**Proof of Theorem ...**

By using Lemmas ..., ..., and ..., it holds that the sequence of processes \( (\mu^n_n, M^n_n, X^n_n) \) is tight on the space \( \mathbb{V}_\times C_{S'} \times C \). Then by the Skorokhod representation theorem, there exists a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), a sequence \( \tau^n = (\tau^n, \mathcal{M}^n, X^n) \) and \( \tau = (\tau, \mathcal{M}, X) \) defined on this space such that:

(i) for each \( n \in \mathbb{N} \), law\( (\gamma^n) = \text{law}(\gamma^n) \),

(ii) there exists a subsequence \( (\tau^n)^k \) of \( (\tau^n) \), still denoted \( (\tau^n) \), which converges to \( \tau \), \( \mathbb{P} \)-a.s. on the space \( \Gamma \).

This means in particular that the sequence of relaxed controls \( (\overline{\tau}^n) \) converges in the weak topology to \( \overline{\tau} \), \( \mathbb{P} \)-a.s. and \( (\overline{\mathcal{M}}^n, \overline{X}^n) \) converges to \( (\overline{\mathcal{M}}, \overline{X}) \), \( \mathbb{P} \)-a.s.in \( C_{S'} \times C \).

According to property (i), we get

\[
\begin{aligned}
\left\{ \begin{array}{l}
X^n_t = x + \int_0^t f_A(s, \overline{X}_s, E(\Psi(\overline{X}_s), a)) \overline{\mu}^n(ds, da) + \int_0^t \int_A \sigma(s, \overline{X}_s, E(\Phi(\overline{X}_s), a) \overline{\mathcal{M}}^n(ds, da),
\end{array} \right.
\end{aligned}
\tag{2.17}
\]

The coefficients \( b, \sigma, \Psi \) and \( \Phi \) being Lipschitz continuous in \( (x,y) \), then according to property (ii) and using similar arguments as in [25] page 32, it holds that

\[
\int_0^t \int_A b(s, \overline{X}_s, E(\Psi(\overline{X}_s), u)) \overline{\mu}^n(ds, da) \text{ converges in probability to } \int_0^t \int_A b(s, \overline{X}_s, E(\Psi(\overline{X}_s), u) \overline{\mu}_s(da)ds
\]

and

\[
\int_0^t \int_A \sigma(s, \overline{X}_s, E(\Phi(\overline{X}_s), a) \overline{\mathcal{M}}^n(ds, da) \text{ converges in probability to } \int_0^t \int_A \sigma(s, \overline{X}_s, E(\Phi(\overline{X}_s), a) \overline{\mathcal{M}}(ds, da).
\]

Therefore \( \overline{X} \) satisfies the MFSDE
\[
\begin{align*}
\begin{cases}
X_t &= x + \int_0^t \int_A b(s, X_s, E(\Psi(X_s)), u) \pi_s(du)ds + \int_0^t \sigma(s, X_s, E(\Psi(X_s))) M(ds, da), \\
X_0 &= x.
\end{cases}
\end{align*}
\tag{2.18}
\]

To finish the proof of Theorem 2.8, it remains to verify that \( \pi \) is an optimal control. According to above properties (i)-(ii) and assumption (H2), we have

\[
\inf_{\mu \in \mathcal{X}} J(\mu) = \lim_{\nu \to \infty} J(\nu),
\]

\[
= \lim_{\nu \to \infty} E \left[ \int_0^T h(t, X^n_t, E(\varphi(X^n_t)), a) \nu_t^n(da)dt + g(X^n_T, E(\lambda(X^n_T))) \right]
\]

\[
= \lim_{\nu \to \infty} E \left[ \int_0^T h(t, X^n_t, E(\varphi(X^n_t)), a) \pi_t^n(da)dt + g(X^n_T, E(\lambda(X^n_T))) \right]
\]

\[
= E \left[ \int_0^T h(t, X_t, E(\varphi(X_t)), a) \pi_t(da)dt + g(X_T, E(\lambda(X_T))) \right].
\]

Hence \( \pi \) is an optimal control. \( \blacksquare \)

The action space \( A \) being compact, we prove that we can restrict the investigation for an optimal relaxed control to the class of so called sliding controls also known as chattering controls. A sliding control is a relaxed control of the form

\[
\mu_t = \sum_{i=1}^p \alpha_i(t) \delta_{u_i(t)}(da), \text{ } u_i(t) \in A, \alpha_i(t) \geq 0 \text{ and } \sum_{i=1}^p \alpha_i(t) = 1. \tag{2.19}
\]

where \( \alpha_i(t) \) and \( u_i(t) \) are stochastic processes.

**Proposition 2.11.** Let \( \mu \) be a relaxed control and \( X \) the corresponding state process. Then one can choose a sliding control

\[
\nu_t = \sum_{i=1}^p \alpha_i(t) \delta_{u_i(t)}(da), \text{ } u_i(t) \in A, \alpha_i(t) \geq 0 \text{ and } \sum_{i=1}^p \alpha_i(t) = 1 \tag{2.20}
\]

such that

1) \( X \) is a solution of the controlled MFSDE

\[
\begin{align*}
\begin{cases}
dX_t &= \sum_{i=1}^p \alpha_i(t) b(t, X_t, E(\Psi(X_t)), u_i(t))dt + \sum_{i=1}^p \alpha_i(t)^{1/2} \sigma(t, X_t, E(\Psi(X_t)), u_i(t))dW^i_t \\
X_0 &= x
\end{cases}
\end{align*}
\tag{2.21}
\]

where \((W^i)\) are independent Brownian motions defined on an extension of the probability space.

**Proposition 2.12.** 2) \( J(\mu) = J(\nu) \).

**Proof.**

Let \( A \) denote the \( d+d^2+1 \)-dimensional simplex

\[
A = \left\{ \lambda = (\lambda_0, \lambda_1, ..., \lambda_{d+d^2+1}) ; \lambda_i \geq 0; \sum_{i=0}^{d+d^2+1} \lambda_i = 1 \right\}
\]

and \( W \) the \( (d+d^2+2) \)-cartesian product of the set \( A \)

\[
W = \{ w = (u_0, u_1, ..., u_{d+d^2+1}) ; \text{ } u_i \in A \}
\]

11
Define the function

\[
g(t, \lambda, w) = \sum_{i=0}^{d+d^2+1} \lambda_i \tilde{b}(t, X_t, E(\Psi(X_t), u_i) - \int_A \tilde{b}(t, X_t, E(\Psi(X_t), a) \mu_t(da)
\]

where \( t \in [0, T], \lambda \in \Lambda, w \in W \) and \( \tilde{b}(t, X_t, E(\Psi(X_t), u_i) = \begin{pmatrix} b(t, X_t, E(\Psi(X_t), u_i) \\ \sigma \sigma^*(t, X_t, E(\Phi(X_t), u_i)) \\ h(t, x_t, E(\varphi(X_t), u_i)) \end{pmatrix} \)

Let \( \tilde{b}(t, X_t, E(\Psi(X_t), u_i), i = 0, 1, ..., d+d^2+1 \) be the subset of \( (d+1) \) arbitrary points in \( P(t, X_t) \) where

\[
P(t, X_t) = \{(b(t, X_t, E(\Psi(X_t), a), \sigma \sigma^*(t, X_t, E(\Phi(X_t), a), h(t, X_t, E(\Psi(X_t), a)) ; a \in A) \subset \mathbb{R}^{d+d^2+1}
\]

Then the convex hull of this set is the collection of all points of the form

\[
\sum_{i=0}^{d+d^2+1} \lambda_i \tilde{b}(t, X_t, E(\Psi(X_t), u_i)
\]

If \( \mu \) is a relaxed control, then \( \int_A \tilde{b}(t, X_t, E(\Psi(X_t), a) \mu_t(da) \in \text{Conv} (P(t, X_t)), \) the convex hull of \( P(t, X_t). \)

Therefore it follows from Carathéodory’s Lemma (which says that the convex hull of a \( d \)-dimensional set \( M \) coincides with the union of the convex hulls of \( d+1 \) points of \( M \)), that for each \( (w, t) \in \Omega \times [0, T] \) the equation \( g(t, \lambda, w) = 0 \) admits at least one solution. Moreover the set

\[
\left\{(\omega, \lambda, w) \in \Omega \times \Lambda \times W : \sum_{i=0}^{d+d^2+1} \lambda_i \tilde{b}(t, X_t, E(\Psi(X_t), u_i) = \int_A \tilde{b}(t, x_t, E(\Psi(x_t), a) \mu_t(da) \right\}
\]

is measurable with respect to \( \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^{d+1}) \otimes \mathcal{B}(\mathcal{A}^{d+1}) \) with non empty \( \omega \)-sections for each \( \omega \).

Hence by using a selection theorem [13], there exist measurable \( \mathcal{F}_t\)–adapted processes \( \lambda_t \) and \( w_t \) with values, respectively in \( \Lambda \) and \( W \) such that:

\[
\int_A \tilde{b}(t, X_t, E(\Psi(X_t), a) \mu_t(du) = \sum_{i=0}^{d+d^2+1} \lambda_i(t) \tilde{b}(t, x_t, u_i(t))
\]

This implies in particular that

\[
\int_A b(t, X_t, E(\Psi(X_t), a) \mu_t(du) = \sum_{i=0}^{d+d^2+1} \lambda_i(t) b(t, X_t, E(\Psi(X_t), u_i(t))
\]

\[
\int_A \sigma \sigma^*(t, X_t, E(\Psi(X_t), a) \mu_t(du) = \sum_{i=0}^{d+d^2+1} \lambda_i(t) \sigma \sigma^*(t, X_t, E(\Psi(X_t), u_i(t))
\]

\[
\int_A h(t, X_t, E(\Psi(X_t), a) \mu_t(du) = \sum_{i=0}^{d+d^2+1} \lambda_i(t) h(t, X_t, E(\Psi(X_t), u_i(t))
\]

Then it is easy to verify that the law of the process defined by the drift \( \sum_{i=0}^{d+d^2+1} \lambda_i(t) b(t, X_t, E(\Psi(X_t), u_i(t)) \)

and its quadratic variation \( \sum_{i=0}^{d+d^2+1} \lambda_i(t) \sigma \sigma^*(t, X_t, E(\Psi(X_t), u_i(t)) \) is the solution of the MFSDE...defined

(possibly on an extension of the initial probability space because of the possible degeneracy of the matrix \( \sigma \sigma^* \)).

This ends the proof.
Corollary 2.13. Assume that the set
\[
P(t, X_t) = \{(b(t, X_t, E(\Psi(X_t), a)), h(t, X_t, E(\Phi(X_t), a)); a \in A) \subset \mathbb{R}^{d+d^2+1}\}
\]
is convex. Then the relaxed optimal control is realized by a strict control.

Proof.
The proof is a direct consequence of Proposition .... Indeed by mimicking the proof of Proposition..., it follows that for each relaxed control \(\tilde{u}(t, X_t)\) we have
\[
\int_A b(t, X_t, E(\Psi(X_t), a)) \mu(d a) \in \text{Conv} (P(t, X_t))
\]
Since \(P(t, X_t)\) is convex then
\[
\text{Conv} (P(t, X_t)) = P(t, X_t)
\]
Then applying the same arguments, there exists a measurable \(\mathcal{F}_t\)-adapted process \(u(t)\) such that
\[
\begin{align*}
\int_A b(t, X_t, E(\Psi(X_t), a)) \mu_t(d a) &= b(t, X_t, u(t)) \\
\int_A \sigma^*(t, X_t, E(\Psi(X_t), a)) \mu_t(d a) &= \sigma^*(t, X_t, E(\Phi(X_t), u(t))) \\
\int_A h(t, X_t, E(\Phi(X_t), a)) \mu_t(d a) &= h(t, X_t, u(t))
\end{align*}
\]
which implies that \(X_t\) is a solution of the MFSDE
\[
\begin{align*}
dX_t &= b(t, X_t, E(\Psi(X_t), u(t)))dt + \sigma(t, X_t, E(\Phi(X_t)))dW_t \\
X_0 &= x
\end{align*}
\]
and \(J(\mu) = J(u)\). This ends the proof.

References


