Dynamics of free-surface and groundwater flows in sandy beaches

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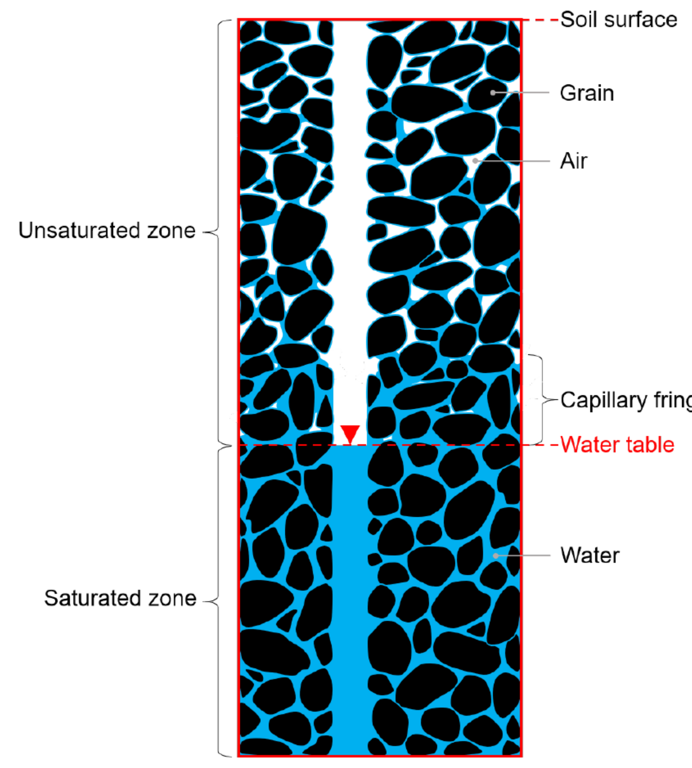
Free surface and **porous media** flows were usually studied separately, but both are essential for **infiltration** problems. This study develops a **coupled model** using the **Shallow Water Equations** and **Richards Equation**, solved with **Discontinuous Galerkin methods** in the code `RIVAGE`. Key innovations include **new methods for penalty calibration and managing wet-dry fronts**, with validation through benchmarks.

Groundwater flow

Richards' Equation (RE) is a *degenerate nonlinear parabolic* equation to describe flow in variably-saturated porous media [2].

$$\partial_t \theta(\psi) - \nabla \cdot (\mathbb{K}(\psi)\nabla h) = 0$$



- h : hydraulic head $[L]$
- z : elevation $[L]$
- $\psi = h - z$: pressure head $[L]$
- $-\mathbb{K}(\psi)\nabla h$: Darcy's speed $[L \cdot T^{-1}]$
- θ : water content $[-]$
- $\mathbb{K}$: hydraulic conductivity $[L \cdot T^{-1}]$

↪ Studied the convergence of the non-linear solver in `RIVAGE`

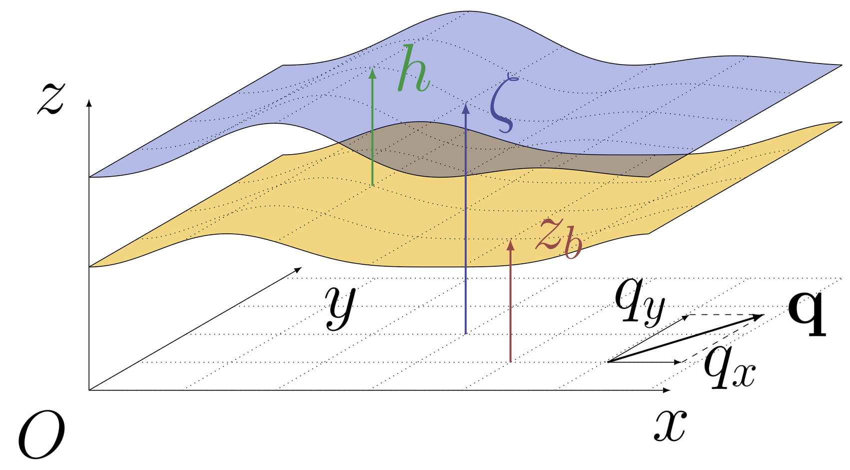
↪ Developed a new way to auto-calibrate penalization parameters of the IIPG formulation

Free-surface flow

Shallow Water Equations (SWE) is a *nonlinear hyperbolic* system of equations to describe free-surface flows under shallow water hypothesis [1].

$$\begin{cases} \partial_t h + \operatorname{div}(\mathbf{q}) = 0, \\ \partial_t \mathbf{q} + \operatorname{div}\left(\dfrac{\mathbf{q} \otimes \mathbf{q}}{h} + \dfrac{gh^2}{2}\mathbb{I}\right) = -gh\nabla z_b \end{cases}$$

- h : water height $[m]$
- z_b : bathymetry elevation $[m]$
- $\zeta = h + z_b$: free surface elevation $[m]$
- $\mathbf{q} = (q_x, q_y)^T$: horizontal discharge $[m^2.s^{-1}]$



↪ Implemented in `RIVAGE` with a *Discontinuous Galerkin method* for solving SWE

↪ Developed a new way to treat wetting and drying fronts

Coupling RE and SWE

Coupled model with *Richards' Equation* and *Shallow Water Equations* to simulate the dynamics of free-surface and groundwater flows in sandy beaches. It is derived folloming the derivation of *Shallow Water Equations with proper boundary conditions at the ground interface*.

$$\begin{cases} I = \mathbf{u}_g \cdot (-\partial_x z_b, -\partial_y z_b, 1)^T, & \text{in } \Omega_{\text{swe}}, \\ \partial_t h + \operatorname{div}(\mathbf{q}) = I, & \text{in } \Omega_{\text{swe}}, \\ \partial_t \mathbf{q} + \operatorname{div}\left(\dfrac{\mathbf{q} \otimes \mathbf{q}}{h} + g\dfrac{h^2}{2}\mathbb{I}\right) = -k(\dfrac{\mathbf{q}}{h})\dfrac{\mathbf{q}}{h} + \dfrac{\alpha_{\text{BJ}}}{\sqrt{k}}(\dfrac{\mathbf{q}}{h} - \mathbf{u}_g) + I\dfrac{\mathbf{q}}{h} - gh\nabla z_b, & \text{in } \Omega_{\text{swe}}, \\ \partial_t \theta(\psi_g) + \operatorname{div}(\mathbf{u}_g) = 0 \text{ with } \mathbf{u}_g = -\mathbb{K}(\psi_g)\nabla h_g, & \text{in } \Omega_g, \\ h_g = h + z_b, & \text{on } \Gamma_C, \end{cases}$$

- Information from RE to SWE treated as a source term
- Information from SWE to RE treated as a boundary condition

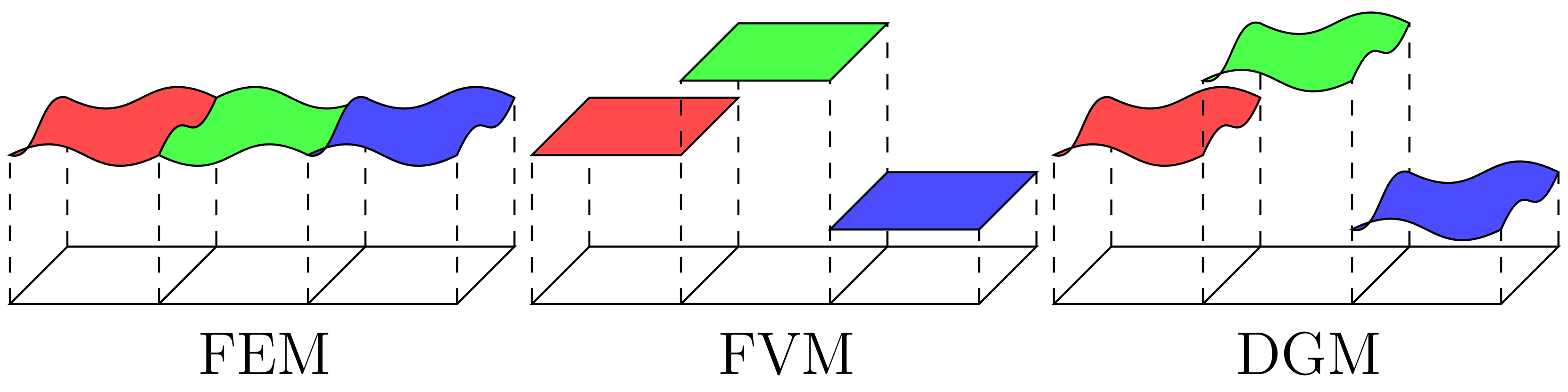
↪ Implemented in `RIVAGE` the two way coupling solved with *Discontinuous Galerkin methods*

↪ Validation of `RIVAGE` on the Steenhauer's test case

Discontinuous Galerkin method

Discontinuous Galerkin methods are a set of *space discretization* methods [3] :

- Based on a *variational formulation* as in *Finite Element Methods* (FEM)
- Designed in an *element-wise* way as in *Finite Volume Methods* (FVM)



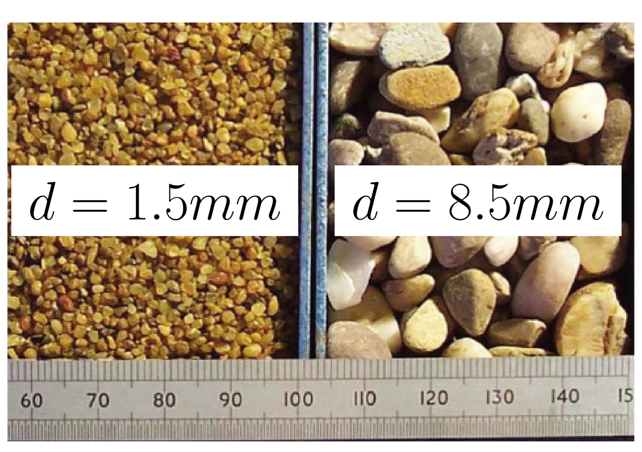
Motivation	Drawbacks
• Adaptive mesh refinement • Non-conformal mesh • Well-suited for solving RE and SWE with any mesh	• High number of degrees of freedom • Penalization parameters with RE • Difficulties to handle dry/wet transitions with SWE

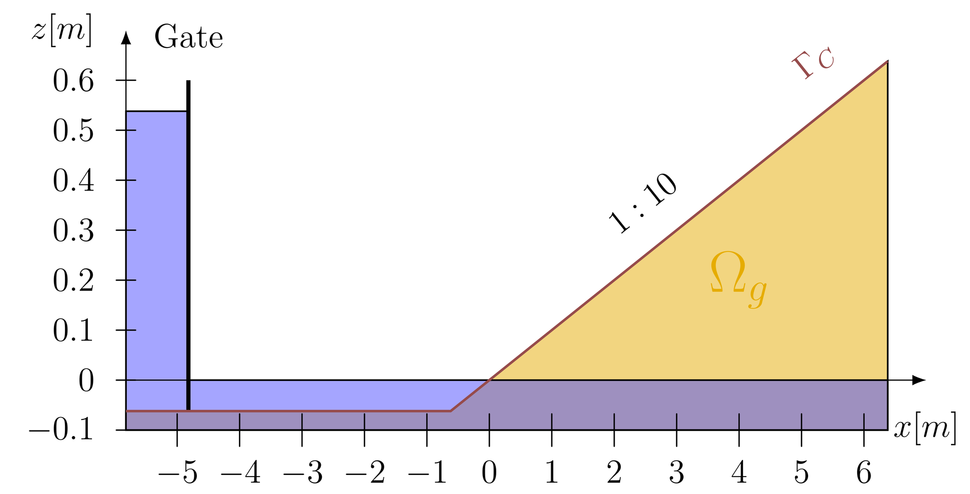
The solution is sought in the space of *piecewise polynomial functions*

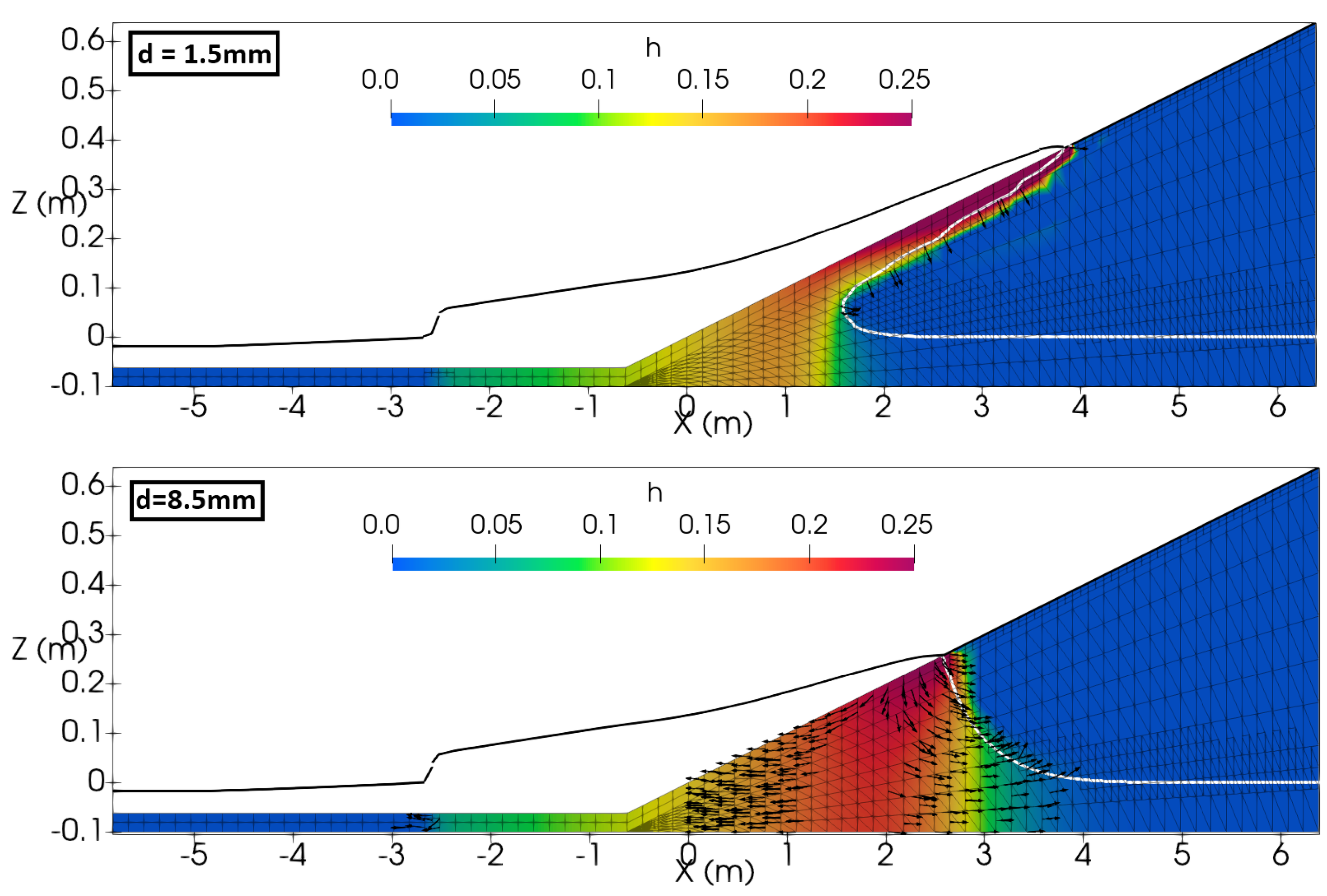
Steenhauer test case

Swash of a wave on a coarse grained beach [4] :

- Two different sediments :
  $K_{1.5} = 1.27\,10^{-2}\ m/s$ and $K_{8.5} = 2.45\,10^{-1}\ m/s$







	Loi constitutive	$\mathbf{K_{s,x}}$	$\mathbf{K_{s,z}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{r}$	$\mathbf{C}_{\text{lam}}$	$\mathbf{C}_{\text{tur}}$
	-	Conductivité hydraulique		Teneur en eau		Coefficients de friction	
$\mathbf{d_{1.5}}$	Relations de Vachaud	$1.27\,10^{-2}$	$2.54\,10^{-3}$	0.3	0.0	0.0	0.01
$\mathbf{d_{8.5}}$	Relations de Vachaud	$2.45\,10^{-1}$	$4.90\,10^{-2}$	0.3	0.0	0.0	0.02

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